# Sheet 2 - solution

1 A lossless transmission line is terminated with a 100 Ω load. If the SWR on the line is 1.5, find the two possible values for the characteristic impedance of the line.

$$|\Gamma| = \frac{S-1}{S+1} = \frac{0.5}{2.5} = 0.2$$

$$|\Gamma| = \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| = \left|\frac{100 - Z_0}{100 + Z_0}\right| \qquad (Z_0 \text{ real})$$

So either, $\frac{100 - Z_0}{100 + Z_0} = 0.2 \Rightarrow Z_0 = Z_L \frac{1-\Gamma}{1+\Gamma} = 100\left(\frac{.8}{1.2}\right) = 66.7\,\Omega$ ✓

or

$\frac{100 - Z_0}{100 + Z_0} = -0.2 \Rightarrow Z_0 = Z_L \frac{1-\Gamma}{1+\Gamma} = 100\left(\frac{1.2}{.8}\right) = 150\,\Omega$ ✓

2 Let $Z_{sc}$ be the input impedance of a length of coaxial line when one end is short-circuited and let $Z_{oc}$ be the input impedance of the line when one end is open-circuited. Derive an expression for the characteristic impedance of the cable in terms of $Z_{sc}$ and $Z_{oc}$.

$Z_{sc} = jZ_0 \tan\beta l \qquad , Z_{oc} = -jZ_0 \cot\beta l$

$Z_{sc} \cdot Z_{oc} = Z_0^2 \Rightarrow Z_0 = \sqrt{Z_{sc} Z_{oc}}$

3 A 100 Ω transmission line has an effective dielectric constant of 1.65. Find the shortest open-circuited length of this line that appears at its input as a capacitor of 5 pF at 2.5 GHz. Repeat for an inductance of 5 nH.

C: $Z_{oc} = -j/\omega C = -j12.73\,\Omega = -jZ_0 \cot\beta l \qquad C = 5pF$

$\tan\beta l = 100/12.73 \Rightarrow \beta l = 82.74^\circ$ ✓

$\lambda_0 = 0.12m$ , $\beta = 2\pi\sqrt{\epsilon_e}/\lambda_0 = 3854^\circ/m \Rightarrow l = \underline{2.147cm}$ ✓

L: $Z_{oc} = j\omega L = +j\,78.5\,\Omega = -jZ_0 \cot\beta l \qquad L = 5nH$

$\tan\beta l = -100/78.5 \Rightarrow \beta l = 128.1^\circ$ ✓ $\Rightarrow l = \underline{3.324}$ cm ✓

These results were verified with Serenade.

4 A radio transmitter is connected to an antenna having an impedance 80 + j40 Ω with a 50 Ω coaxial cable. If the 50 Ω transmitter can deliver 30 W when connected to a 50 Ω load, how much power is delivered to the antenna?

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{30 + j40}{130 + j40} = \frac{50\angle 53^\circ}{136\angle 17^\circ} = 0.367\angle 36^\circ \checkmark$$

$$P_{LOAD} = P_{INC} - P_{REF} = P_{INC}\left(1 - |\Gamma|^2\right) = 30\left[1 - (.367)^2\right] = 25.9\ W \checkmark$$

5 The transmission line circuit shown below has $V_g$= 15 V rms, $Z_g$=75 Ω, $Z_o$ =75 Ω, $Z_L$= 60- j40Ω, and $\ell$ = 0.7λ. Compute the power delivered to the load using three different techniques:

(a) find Γ and compute

$$P_L = \left(\frac{V_g}{2}\right)^2 \frac{1}{Z_0}\left(1 - |\Gamma|^2\right);$$

$V_g = 15\text{v RMS}$, $Z_g = 75\Omega$, $Z_o = 75\Omega$, $Z_L = 60 - j40\Omega$, $\ell = 0.7\lambda$.

a) $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{-15 - j40}{135 - j40} = \frac{42.7\angle -110.6^\circ}{140.8\angle -16.5^\circ} = 0.303\angle -94^\circ = -0.021 - j0.302$$

$$P_L = \left(\frac{V_g}{2}\right)^2 \frac{1}{Z_o}\left(1 - |\Gamma|^2\right) = 0.681\ W \checkmark$$

This method is actually based on $P_L = P_{inc}(1 - |\Gamma|^2)$. It is the simplest method, but only applies to lossless lines.

(b) find $Z_{in}$ and compute

$$P_L = \left|\frac{V_g}{Z_g + Z_{in}}\right|^2 \text{Re}(Z_{in}); \text{ and}$$

b) $$Z_{in} = Z_o \frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell} = 75\,\frac{60 + j190.8}{198.1 + j184.7} = 75\,\frac{200\angle 72.5^\circ}{270.8\angle 43^\circ}$$

$$= 55.4\angle 29.5^\circ = 48.2 + j27.3\ \Omega$$

$$P_L = \left|\frac{V_g}{Z_g + Z_{in}}\right|^2 Re(Z_{in}) = \left|\frac{15}{123.2 + j27.3}\right|^2 (48.2) = 0.681\,W \checkmark$$

This method computes $P_L = P_{in} = |I_{in}|^2 R_{in}$, and also applies only to lossless lines.

(c) find $V_L$ and compute

$$P_L = \left|\frac{V_L}{Z_L}\right|^2 \mathrm{Re}(Z_L).$$

$$V(z) = V^+(e^{-j\beta z} + \Gamma e^{j\beta z})$$

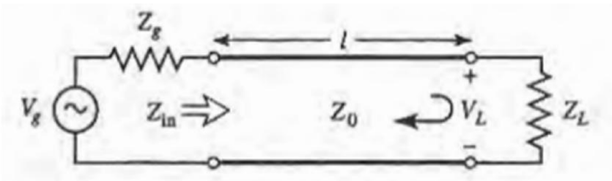
$$V_L = V(0) = V^+(1+\Gamma) \qquad V^+ = \frac{V_g}{2} = 7.5\,\mathrm{v}$$

$$= 7.5(1 - .021 - j.302)$$

$$= 7.68 \angle{-17^\circ}$$

$$P_L = \left|\frac{V_L}{Z_L}\right|^2 Re(Z_L) = \left(\frac{7.68}{72.1}\right)^2 (60) = 0.681\,\mathrm{w} \checkmark$$

(d) Discuss the rationale for each of these methods. Which of these methods can be used if the line is not lossless?




This method computes $P_L = |I_L|^2 R_L$, and applies to lossy as well as lossless lines. Note the concept that $V^+ = V_g/2$ requires a good understanding of the transmission line equations, and only applies here because $Z_g = Z_0$.

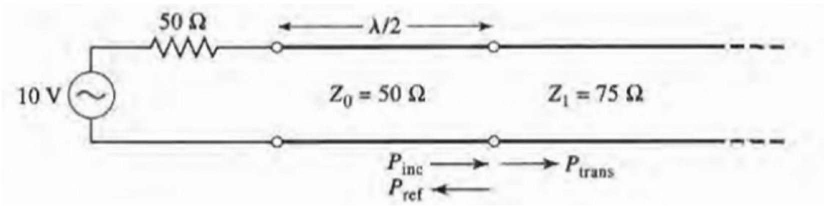
$\boxed{6}$ For a purely reactive load impedance of the form $Z_L = jX$, show that the reflection coefficient magnitude lf I Γ I is always unity. Assume the characteristic impedance $Z_o$ is real.

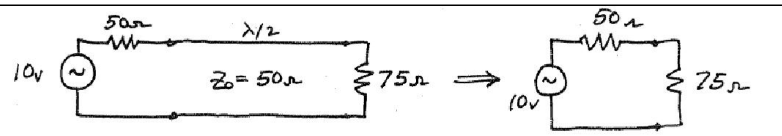
$$Z_L = jX$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX - Z_0}{jX + Z_0}$$

$$|\Gamma|^2 = \Gamma\Gamma^* = \frac{(jX - Z_0)}{(jX + Z_0)}\frac{(-jX - Z_0)}{(-jX + Z_0)} = \frac{X^2 - j\cancel{Z_0 X} + j\cancel{Z_0 X} + Z_0^2}{X^2 + Z_0^2} = 1 \checkmark$$

$\boxed{7}$ Consider the transmission line circuit shown below. Compute the incident power, the reflected power, and the power transmitted into the infinite 75 Ω line. Show that power conservation is satisfied.




POWER DELIVERED BY SOURCE $= \frac{1}{2}\frac{(10)^2}{50+75} = 0.400$ W ✓

POWER DISSIPATED IN 50Ω LOAD $= \frac{1}{2}(50)\left(\frac{10}{50+75}\right)^2 = 0.160$ W ✓

POWER TRANSMITTED DOWN LINE $= \frac{1}{2}(75)\left(\frac{10}{50+75}\right)^2 = 0.240$ W ✓

INCIDENT POWER $= \frac{1}{2}(50)\left(\frac{10}{50+50}\right)^2 = 0.250$ W ✓

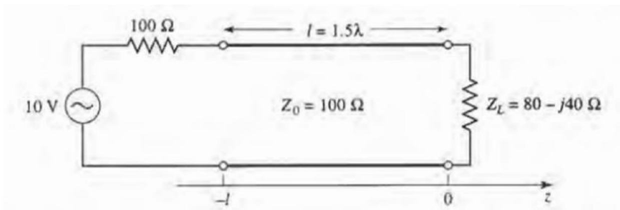
REFLECTED POWER $= P_{INC}\,|\Gamma|^2 = .250\left|\frac{75-50}{75+50}\right|^2 = 0.010$ W ✓

$P_{INC} - P_{REF} = .250 - .010 = 0.240 = P_{TRANS}$ ✓

$P_{DISS} + P_{TRANS} = .160 + .240 = 0.400 = P_{SOURCE}$ ✓

$\boxed{8}$ A generator is connected to a transmission line as shown below. Find the voltage as a function of z along the transmission line. Plot the magnitude of this voltage for

$-\ell \leq Z \leq 0$



$$\Gamma = \frac{-20-j40}{180-j40} = \frac{44.7\angle{-116.6^\circ}}{184.4\angle{-12.5^\circ}} = 0.24\angle{-104^\circ} = -0.058 - j\,0.233 \checkmark$$

$$V_L = 10\,\frac{80-j40}{180-j40} = 10\,\frac{89.4\angle{-26^\circ}}{184\angle{-12.5^\circ}} = 4.86\angle{-13.5^\circ}$$

$$V(z) = V^+\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right] \qquad V^+ = 10\,\frac{100}{100+100} = 5\,v \checkmark$$

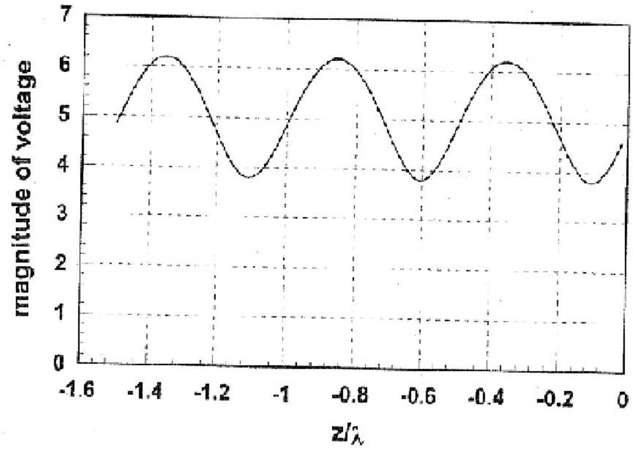
So $$V(z) = 5\left[e^{-j\beta z} + \Gamma e^{j\beta z}\right]$$

$V_{MAX} = 5(1+|\Gamma|) = 5(1.24) = 6.2$ at $z = -0.355\lambda$

$V_{MIN} = 5(1-|\Gamma|) = 5(.76) = 3.8$ at $z = -0.105\lambda$

These results repeat every $\lambda/2$.

$|V(z)|$ is plotted below




Good Luck

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